

# Anisotropic Hardy inequalities

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**ABSTRACT.** We study some Hardy-type inequalities involving a general norm in  $\mathbb{R}^n$  and an anisotropic distance function to the boundary. The case of the optimality of the constants is also addressed.

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## 1 INTRODUCTION

Let  $F$  be a smooth norm of  $\mathbb{R}^n$ . In this paper we investigate the validity of Hardy-type inequalities

$$\int_{\Omega} F(\nabla u)^2 \, dx \geq C_F(\Omega) \int_{\Omega} \frac{u^2}{d_F^2} \, dx, \quad \forall u \in H_0^1(\Omega), \quad (1.1)$$

where  $\Omega$  is a domain of  $\mathbb{R}^n$ , and  $d_F$  is the anisotropic distance to the boundary with respect the dual norm (see Section 2 for the

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precise assumptions and definitions). We aim to study the best possible constant for which (1.1) holds, in the sense that

$$C_F(\Omega) = \inf_{u \in H_0^1(\Omega)} \frac{\int_{\Omega} F(\nabla u)^2 \, dx}{\int_{\Omega} \frac{u^2}{d_F^2} \, dx}.$$

In the case of the Euclidean norm, that is when  $F = \mathcal{E} = |\cdot|$ , the inequality (1.1) reduces to

$$\int_{\Omega} |\nabla u|^2 \, dx \geq C_{\mathcal{E}}(\Omega) \int_{\Omega} \frac{u^2}{d_{\mathcal{E}}^2} \, dx, \quad \forall u \in H_0^1(\Omega), \quad (1.2)$$

where

$$d_{\mathcal{E}}(x) = \inf_{y \in \partial\Omega} |x - y|, \quad x \in \Omega$$

is the usual distance function from the boundary of  $\Omega$ , and  $C_{\mathcal{E}}$  is the best possible constant.

The inequality (1.2) has been studied by many authors, under several points of view. For example, it is known that for any bounded domain with Lipschitz boundary  $\Omega$  of  $\mathbb{R}^n$ ,  $0 < C_{\mathcal{E}}(\Omega) \leq \frac{1}{4}$  (see [D95, MMP98, BM97]). In particular, if  $\Omega$  is a convex domain of  $\mathbb{R}^n$ , the optimal constant  $C_{\mathcal{E}}$  in (1.2) is independent of  $\Omega$ , and its value is  $C_{\mathcal{E}} = \frac{1}{4}$ , but there are smooth bounded domains such that  $C_{\mathcal{E}}(\Omega) < \frac{1}{4}$  (see [MMP98, MS97]). Furthermore, in [MMP98] it is proved that  $C_{\mathcal{E}}$  is achieved if and only if it is strictly smaller than  $\frac{1}{4}$ .

Actually, the value of the best constant  $C_{\mathcal{E}}(\Omega)$  is still  $\frac{1}{4}$  for a more general class of domains. This has been shown, for example, in [BFT04], under the assumption that  $d_{\mathcal{E}}$  is superharmonic in  $\Omega$ , in the sense that

$$\Delta d_{\mathcal{E}} \leq 0 \quad \text{in } \mathcal{D}'(\Omega). \quad (1.3)$$

As proved in [LLL12], when  $\Omega$  is a  $C^2$  domain the condition (1.3) is equivalent to require that  $\partial\Omega$  is weakly mean convex, that is its mean curvature is nonnegative at any point.

The fact that the constant  $C_{\mathcal{E}}(\Omega) = \frac{1}{4}$  is not achieved has lead to the interest of studying “improved” versions of (1.2), by adding remainder terms which depend, in general, on suitable norms of  $u$ . For instance, if  $\Omega$  is a bounded and convex set, in [BM97] it has been showed that

$$\int_{\Omega} |\nabla u|^2 \, dx \geq \frac{1}{4} \int_{\Omega} \frac{u^2}{d_{\mathcal{E}}^2} \, dx + \frac{1}{4L^2} \int_{\Omega} u^2 \, dx, \quad \forall u \in H_0^1(\Omega), \quad (1.4)$$

where  $L$  is the diameter of  $\Omega$ . Actually, in [HHL02] the authors showed that the value  $\frac{1}{4L^2}$  can be replaced by a constant which depends on the volume of  $\Omega$ , namely  $c(n)|\Omega|^{-\frac{2}{n}}$ ; here  $c(n)$  is suitable constant depending only on the dimension of the space (see also [FMT06]).

When  $\Omega$  satisfies condition (1.3), several improved versions of (1.2) can be found for instance in [BFT03, BFT04, FMT06]. More precisely, in [BFT04] the authors proved that

$$\int_{\Omega} |\nabla u|^2 dx - \frac{1}{4} \int_{\Omega} \frac{|u|^2}{d_{\varepsilon}^2} dx \geq \frac{1}{4} \int_{\Omega} \frac{|u|^2}{d_{\varepsilon}^2} \left( \log \frac{D_0}{d_{\varepsilon}} \right)^{-2} dx, \quad (1.5)$$

where  $D_0 \geq e \cdot \sup\{d_{\varepsilon}(x, \partial\Omega)\}$  and  $u \in H_0^1(\Omega)$ .

The aim of this paper is to study Hardy inequalities of the type (1.1), and to show improved versions in the anisotropic setting given by means of the norm  $F$ , in the spirit of (1.5). For example, one of our main result states that for suitable domains  $\Omega$  of  $\mathbb{R}^n$ , and for every function  $u \in H_0^1(\Omega)$ , it holds that

$$\int_{\Omega} F^2(\nabla u) dx - \frac{1}{4} \int_{\Omega} \frac{u^2}{d_F^2} dx \geq \frac{1}{4} \int_{\Omega} \frac{u^2}{d_F^2} \left( \log \frac{D}{d_F} \right)^{-2} dx, \quad (1.6)$$

where  $D = e \cdot \sup\{d_F(x, \partial\Omega), x \in \Omega\}$ .

The condition we will impose on  $\Omega$  in order to have (1.6) will involve the sign of an anisotropic Laplacian of  $d_F$  (see sections 2 and 3). We will also show that such condition is, in general, not equivalent to (1.3).

Actually, we will deal also with the optimality of the involved constants. Moreover, we will show that (1.6) implies an improved version of (1.1) in terms of the  $L^2$  norm of  $u$ , in the spirit of (1.4). More precisely, we will show that if  $\Omega$  is a convex bounded open set then

$$\int_{\Omega} F(\nabla u)^2 dx - \frac{1}{4} \int_{\Omega} \frac{|u|^2}{d_F^2} dx \geq C(n)|\Omega|^{-\frac{2}{n}} \int_{\Omega} |u|^2 dx.$$

We emphasize that Hardy type inequalities in anisotropic settings have been studied, for example, in [V06, DPG15, BF04], where, instead of considering the weight  $d_F^{-2}$ , it is taken into account a function of the distance from a point of the domain (see for example [BFT04, GP98, BV97, HLP, VZ00, AFV12, BCT07] and the reference therein for the Euclidean case).

The structure of the paper is the following. In Section 2 we fix the necessary notation and provide some preliminary results

which will be needed later. Moreover, we discuss in some details the condition we impose on  $\Omega$  in order to be (1.6) and (1.1) true. In Section 3 we study the inequality (1.1) and we give some applications. In Section 4 the improved versions of (1.1) are investigated and, finally, Section 5 is devoted to the study of the optimality of the constants in (1.6).

## 2 NOTATION AND PRELIMINARIES

Throughout the paper we will consider a convex even 1-homogeneous function

$$\xi \in \mathbb{R}^n \mapsto F(\xi) \in [0, +\infty[,$$

that is a convex function such that

$$F(t\xi) = |t|F(\xi), \quad t \in \mathbb{R}, \xi \in \mathbb{R}^n, \quad (2.1)$$

and such that

$$\alpha_1 |\xi| \leq F(\xi), \quad \xi \in \mathbb{R}^n, \quad (2.2)$$

for some constant  $0 < \alpha_1$ . Under this hypothesis it is easy to see that there exists  $\alpha_2 \geq \alpha_1$  such that

$$H(\xi) \leq \alpha_2 |\xi|, \quad \xi \in \mathbb{R}^n.$$

Furthermore we suppose that  $F^2$  is strongly convex, in the sense that  $F \in C^2(\mathbb{R}^n \setminus \{0\})$  and

$$\nabla_{\xi}^2 F^2 > 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

In this context, an important role is played by the polar function of  $F$ , namely the function  $F^\circ$  defined as

$$x \in \mathbb{R}^n \mapsto F^\circ(x) = \sup_{\xi \neq 0} \frac{\xi \cdot x}{F(\xi)}.$$

It is not difficult to verify that  $F^\circ$  is a convex, 1-homogeneous function that satisfies

$$\frac{1}{\alpha_2} |\xi| \leq F^\circ(\xi) \leq \frac{1}{\alpha_1} |\xi|, \quad \forall \xi \in \mathbb{R}^n. \quad (2.3)$$

Moreover, the hypotheses on  $F$  ensures that  $F^\circ \in C^2(\mathbb{R}^n \setminus \{0\})$  (see for instance [S93])

$$F(x) = (F^\circ)^\circ(x) = \sup_{\xi \neq 0} \frac{\xi \cdot x}{F^\circ(\xi)}.$$

The following well-known properties hold true:

$$F_\xi(\xi) \cdot \xi = F(\xi), \quad \xi \neq 0, \quad (2.4)$$

$$F_\xi(t\xi) = \text{sign } t \cdot F_\xi(\xi), \quad \xi \neq 0, t \neq 0, \quad (2.5)$$

$$\nabla_\xi^2 F(t\xi) = \frac{1}{|t|} \nabla_\xi^2 F(\xi) \quad \xi \neq 0, t \neq 0, \quad (2.6)$$

$$F(F_\xi^0(\xi)) = 1, \quad \forall \xi \neq 0, \quad (2.7)$$

$$F^0(\xi) F_\xi(F_\xi^0(\xi)) = \xi \quad \forall \xi \neq 0. \quad (2.8)$$

Analogous properties hold interchanging the roles of  $F$  and  $F^0$ .

The open set

$$\mathcal{W} = \{\xi \in \mathbb{R}^n : F^0(\xi) < 1\}$$

is the so-called Wulff shape centered at the origin. More generally, we denote

$$\mathcal{W}_r(x_0) = r\mathcal{W} + x_0 = \{x \in \mathbb{R}^2 : F^0(x - x_0) < r\},$$

and  $\mathcal{W}_r(0) = \mathcal{W}_r$ .

We recall the definition and some properties of anisotropic curvature for a smooth set. For further details we refer the reader, for example, to [ATW93] and [BP96].

**Definition 2.1.** Let  $A \subset \mathbb{R}^n$  be an open set with smooth boundary. The anisotropic outer normal  $n_A$  is defined as

$$n_A(x) = \nabla_\xi F(v_A(x)), \quad x \in \partial A,$$

where  $v_A$  is the unit Euclidean outer normal to  $\partial A$ .

**Remark 2.1.** We stress that if  $A = \mathcal{W}_r(x_0)$ , by the properties of  $F$  it follows that

$$n_A(x) = \nabla_\xi F(\nabla_\xi F^0(x - x_0)) = \frac{1}{r}(x - x_0), \quad x \in \partial A.$$

Finally, let us recall the definition of Finsler Laplacian

$$\Delta_F u = \text{div}(F(\nabla u) F_\xi(\nabla u)), \quad (2.9)$$

defined for  $u \in H_0^1(\Omega)$ .

**Definition 2.2.** Let  $u \in H_0^1(\Omega)$ , we say that  $u$  is  $\Delta_F$ -superharmonic if

$$-\Delta_F u \geq 0 \quad \text{in } \mathcal{D}', \quad (C_H)$$

that is

$$\int_{\Omega} F(\nabla u) F_\xi(\nabla u) \cdot \nabla \varphi \, dx \geq 0, \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0.$$

Similarly, by writing that  $u$  is  $\Delta$ -superharmonic we will mean that  $u$  is superharmonic in the usual sense, that is  $-\Delta u \geq 0$ .

### 2.1 Anisotropic distance function

Due to the nature of the problem, it seems to be natural to consider a suitable notion of distance to the boundary.

Let us consider a domain  $\Omega$ , that is a connected open set of  $\mathbb{R}^n$ , with non-empty boundary.

The anisotropic distance of  $x \in \overline{\Omega}$  to the boundary of  $\partial\Omega$  is the function

$$d_F(x) = \inf_{y \in \partial\Omega} F^0(x - y), \quad x \in \overline{\Omega}. \quad (2.10)$$

We stress that when  $F = |\cdot|$  then  $d_F = d_\varepsilon$ , the Euclidean distance function from the boundary.

It is not difficult to prove that  $d_F$  is a uniform Lipschitz function in  $\overline{\Omega}$  and, using the property (2.7),

$$F(\nabla d_F(x)) = 1 \quad \text{a.e. in } \Omega. \quad (2.11)$$

Obviously, assuming  $\sup_\Omega d_F < +\infty$ ,  $d_F \in W_0^{1,\infty}(\Omega)$  and the quantity

$$r_F = \sup\{d_F(x), x \in \Omega\}, \quad (2.12)$$

is called anisotropic inradius of  $\Omega$ .

For further properties of the anisotropic distance function we refer the reader to [CM07].

The main assumption in this paper will be that  $d_F$  is  $\Delta_F$ -superharmonic in the sense of Definition 2.2.

**Remark 2.2.** We emphasize that if  $\Omega$  is a convex set, the functions  $d_\varepsilon$  and  $d_F$  are respectively  $\Delta$  and  $\Delta_F$ -superharmonic. This can be easily proved by using the concavity of  $d_\varepsilon$  and  $d_F$  in  $\Omega$  (see for instance [EG92], and [DPG14] for the anisotropic case).

Actually, in the Euclidean case there exist non-convex sets for which  $d_\varepsilon$  is still  $\Delta$ -superharmonic. An example can be obtained for instance in dimension  $n = 3$ , taking the standard torus (see [AK85]). Similarly, in the following example we show that there exists a non-convex set such that the anisotropic distance function  $d_F$  is  $\Delta_F$ -superharmonic.

**Example 2.1.** Let us consider the following Finsler norm in  $\mathbb{R}^3$

$$F(x_1, x_2, x_3) = (x_1^2 + x_2^2 + a^2 x_3^2)^{\frac{1}{2}},$$

with  $a > 0$ ; then

$$F^0(x_1, x_2, x_3) = \left( x_1^2 + x_2^2 + \frac{x_3^2}{a^2} \right)^{\frac{1}{2}}.$$

We consider the set  $\Omega \subset \mathbb{R}^3$  obtained by rotating the ellipse

$$\gamma = \{(0, x_2, x_3) : (x_2 - R)^2 + \frac{x_3^2}{a^2} < r^2\} \quad \text{with } R > r > 0,$$

about the  $x_3$ -axis. Obviously  $\Omega$  is not convex. In order to show that  $(C_H)$  holds, we first observe that if we fix a generic point  $x \in \gamma$ , then being  $F$  isotropic with respect to the first two components the anisotropic distance is achieved in a point  $\bar{x}$  of the boundary of  $\gamma$ . Moreover it is not difficult to show that the vector  $\bar{x} - x$  has the same direction of the anisotropic normal  $n_\Omega$  (see Definition 2.1, and see also [DPG13]). Hence by Remark 2.1

$$d_F(x) = r - F^0(x - x_0),$$

where  $x_0 = (0, R, 0)$  is the center of the ellipse.

Now let us introduce a plane polar coordinates  $(\rho, \vartheta)$  such that to a generic point  $Q = (x_1, x_2, x_3) \in \mathbb{R}^3$ , it is associated the point  $Q' = (\rho \cos \vartheta, \rho \sin \vartheta, x_3)$ , where  $\rho = \sqrt{x_1^2 + x_2^2}$  and  $\vartheta \in [0, 2\pi]$ . Then, by construction,

$$\Omega = \{Q' \in \mathbb{R}^3 : F^0(Q' - C) < r\},$$

where  $C = (R \cos \vartheta, R \sin \vartheta, 0)$  and  $F^0(Q' - C)^2 = (R - \rho)^2 + \frac{x_3^2}{a^2}$ .

Then as observed before, fixed  $Q' \in \Omega$

$$\begin{aligned} d_F(Q') &= r - F^0(Q' - C) = \\ &= r - \sqrt{(R - \rho)^2 + \frac{x_3^2}{a^2}} = r - \sqrt{\left(R - \sqrt{x_1^2 + x_2^2}\right)^2 + \frac{x_3^2}{a^2}}. \end{aligned}$$

Now we are in position to prove  $(C_H)$ . We note that for all  $Q' \neq C$

$$\begin{aligned} \Delta_F d_F(Q') &= \operatorname{div}(F(\nabla d_F) F_\xi(\nabla d_F)) \\ &= \frac{\partial^2 d_F}{\partial x_1^2} + \frac{\partial^2 d_F}{\partial x_2^2} + a^2 \frac{\partial^2 d_F}{\partial x_3^2} \\ &= \frac{1}{\rho} \frac{\partial d_F}{\partial \rho} + \frac{\partial^2 d_F}{\partial \rho^2} + a^2 \frac{\partial^2 d_F}{\partial x_3^2} \\ &= \frac{R - 2\rho}{\rho F^0(Q' - C)}. \end{aligned} \tag{2.13}$$

Being  $\rho > R - r$ , we get that  $d_F$  is  $\Delta_F$ -superharmonic in  $\Omega$  if  $R > 2r$  for all  $a > 0$ .

**Remark 2.3.** In general, if  $\Omega$  is not convex, to require that  $d_F$  is  $\Delta_F$ -superharmonic does not assure that  $d_\varepsilon$  is  $\Delta$ -superharmonic. Indeed, let  $\Omega$  be as in Example 2.1; if we take  $R \geq 2r$  then, as showed before,  $-\Delta_F d_F \geq 0$ . On the other hand it is possible to choose  $\alpha > 0$  such that  $d_\varepsilon$  is not  $\Delta$ -superharmonic. To do that, it is enough to prove that the mean curvature of  $\Omega$  is negative at some points of the boundary for a suitable choice of  $\alpha$ . Indeed in [LLL12] it is proved that  $d_\varepsilon$  is  $\Delta$ -superharmonic on  $\Omega$  if and only if the mean curvature  $H_\Omega(y) \geq 0$  for all  $y \in \partial\Omega$ .

The parametric equations of  $\partial\Omega$  are

$$\varphi(t, \vartheta) : \begin{cases} x_1 = (R + r \cos \vartheta) \cos t = \phi(\vartheta) \cos t \\ x_2 = (R + r \cos \vartheta) \sin t = \phi(\vartheta) \sin t \\ x_3 = \alpha r \sin \vartheta = \psi(\vartheta), \end{cases}$$

where  $t, \vartheta \in [0, 2\pi]$ .

Then for  $y = \varphi(t, \vartheta) \in \partial\Omega$  we have

$$\begin{aligned} H_\Omega(y) &= -\frac{\phi(\phi''\psi' - \phi'\psi'') - \psi'((\phi')^2 + (\psi')^2)}{2|\phi|((\phi')^2 + (\psi')^2)^{\frac{3}{2}}} \\ &= \frac{\alpha r^2(R + 2r \cos \vartheta + r \cos^3 \vartheta(\alpha^2 - 1))}{2|R + r \cos \vartheta|(r^2 \sin^2 \vartheta + \alpha^2 r^2 \cos^2 \vartheta)^{\frac{3}{2}}}. \end{aligned} \quad (2.14)$$

Finally we observe that if  $\vartheta = \pi$  then  $H_\Omega(y) < 0$ , if  $\alpha > 1$ .

### 3 ANISOTROPIC HARDY INEQUALITY

**Theorem 3.1.** *Let  $\Omega$  be a domain in  $\mathbb{R}^n$  and suppose that condition (C<sub>H</sub>) holds. Then for every function  $u \in H_0^1(\Omega)$  the following anisotropic Hardy inequality holds*

$$\int_{\Omega} F(\nabla u)^2 \, dx \geq \frac{1}{4} \int_{\Omega} \frac{u^2}{d_F^2} \, dx \quad (3.1)$$

where  $d_F$  is the anisotropic distance function from the boundary of  $\Omega$  defined in (2.10).

*Proof.* First we prove inequality (3.1). Being  $F^2$  convex, we have that

$$F(\xi_1)^2 \geq F(\xi_2)^2 + 2F(\xi_2)F_\xi(\xi_2) \cdot (\xi_1 - \xi_2).$$



Hence putting  $\xi_1 = \nabla u$  and  $\xi_2 = Au \frac{\nabla d_F}{d_F}$ , with  $A$  positive constant, recalling that  $F(\nabla d_F) = 1$ , by the homogeneity of  $F$  we get

$$\int_{\Omega} F(\nabla u)^2 dx \geq -A^2 \int_{\Omega} \frac{u^2}{d_F^2} dx + A \int_{\Omega} \frac{2u}{d_F} F_{\xi}(\nabla d_F) \cdot \nabla u dx.$$

By the Divergence Theorem (in a general setting, contained for example in [A83]) we have

$$\begin{aligned} \int_{\Omega} F(\nabla u)^2 dx &\geq -A^2 \int_{\Omega} \frac{u^2}{d_F^2} dx + A \int_{\Omega} \frac{F_{\xi}(\nabla d_F)}{d_F} \cdot \nabla(u^2) dx \\ &\geq -A^2 \int_{\Omega} \frac{u^2}{d_F^2} dx - A \int_{\Omega} u^2 \frac{\Delta_F d_F}{d_F} dx + A \int_{\Omega} \frac{u^2}{d_F^2} dx \end{aligned}$$

Being  $-\Delta_F d_F \geq 0$  we get

$$\int_{\Omega} F(\nabla u)^2 dx \geq (A - A^2) \int_{\Omega} \frac{u^2}{d_F^2} dx.$$

Then maximizing with respect to  $A$  we obtain that  $A = \frac{1}{2}$ , and then (3.1) follows.  $\square$

**Remark 3.1.** We observe that if  $\Omega$  is a convex domain in  $\mathbb{R}^n$ , an inequality of the type (3.1) can be immediately obtained by using the following classical Hardy inequality involving the Euclidean distance function  $d_{\mathcal{E}}$

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{u^2}{d_{\mathcal{E}}^2} dx. \quad (3.2)$$

By (2.2) we easily get

$$\int_{\Omega} F(\nabla u)^2 dx \geq \alpha^2 \int_{\Omega} |\nabla u|^2 dx \geq \frac{\alpha^2}{4} \int_{\Omega} \frac{u^2}{d_{\mathcal{E}}^2} dx \geq \frac{1}{4} \frac{\alpha_1^2}{\alpha_2^2} \int_{\Omega} \frac{u^2}{d_F^2}, \quad (3.3)$$

where the constant in the right-hand side is smaller than  $\frac{1}{4}$  since  $\alpha_1 < \alpha_2$ . We emphasize that if  $\Omega$  is not convex inequality (3.3) holds under the assumption that  $d_{\mathcal{E}}$  is  $\mathcal{E}$ -superharmonic, since (3.2) is in force. On the other hand the assumption on  $d_{\mathcal{E}}$  is not related with the hypothesis required about  $d_F$  in the Theorem 3.1, as observed in Remark 2.3.

Using Theorem 3.1 it is not difficult to obtain a lower bound for the first eigenvalue of  $\Delta_F$  defined in (2.9).

**Corollary 3.1.** *Let  $\Omega$  be a bounded domain of  $\mathbb{R}^n$  and suppose that condition  $(C_H)$  holds. Let  $\lambda_1(\Omega)$  be the first Dirichlet eigenvalue of the Finsler Laplacian, that is*

$$\lambda_1(\Omega) = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_{\Omega} [F(\nabla u)]^2 dx}{\int_{\Omega} |u|^2 dx}. \quad (3.4)$$

Then

$$\lambda_1(\Omega) \geq \frac{1}{4r_F^2},$$

where  $r_F$  is the anisotropic inradius of  $\Omega$  defined in (2.12).

*Proof.* Let  $v$  the first eigenfunction related to  $\lambda_1(\Omega)$  such that  $\|v\|_{L^2(\Omega)} = 1$ . Then (3.4) and inequality (3.1) imply

$$\lambda_1(\Omega) = \int_{\Omega} [F(\nabla v)]^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{v^2}{d_F^2} dx \geq \frac{1}{4r_F^2},$$

that is the claim.  $\square$

#### 4 HARDY INEQUALITY WITH A REMINDER TERM

**Theorem 4.1.** *Let  $\Omega$  be a domain of  $\mathbb{R}^n$ . Let us suppose also that condition  $(C_H)$  holds, and  $\sup\{d_F(x, \partial\Omega), x \in \Omega\} < +\infty$ . Then for every function  $u \in H_0^1(\Omega)$  the following inequality holds:*

$$\int_{\Omega} F^2(\nabla u) dx - \frac{1}{4} \int_{\Omega} \frac{u^2}{d_F^2} dx \geq \frac{1}{4} \int_{\Omega} \frac{u^2}{d_F^2} \left( \log \frac{D}{d_F} \right)^{-2} dx, \quad (4.1)$$

where  $D = e \cdot \sup\{d_F(x, \partial\Omega), x \in \Omega\}$ .

*Proof.* We will use the following notation:

$$X(t) = -\frac{1}{\log t}, \quad t \in ]0, 1[.$$

Being  $F^2$  convex, we have that

$$F(\xi_1)^2 \geq F(\xi_2)^2 + 2F(\xi_2)F_{\xi}(\xi_2) \cdot (\xi_1 - \xi_2).$$

Let us consider

$$\xi_1 = \nabla u, \quad \xi_2 = \frac{u}{2} \frac{\nabla d_F}{d_F} \left[ 1 - X\left(\frac{d_F}{D}\right) \right].$$

Being  $d_F(x) \leq \frac{D}{e}$ , by the 1-homogeneity of  $F$  we get

$$\begin{aligned}
 F(\nabla u)^2 dx &\geq \\
 &\frac{1}{4} \frac{u^2}{d_F^2} F^2(\nabla d) \left[ 1 - \chi \left( \frac{d_F}{D} \right) \right]^2 + \frac{u}{d_F} \left[ 1 - \chi \left( \frac{d_F}{D} \right) \right] \times \\
 &\times F(\nabla d) F_\xi(\nabla d) \cdot \left( \nabla u - \frac{u}{2} \frac{\nabla d_F}{d_F} \left[ 1 - \chi \left( \frac{d_F}{D} \right) \right] \right) = \\
 &= -\frac{1}{4} \frac{u^2}{d_F^2} \left[ 1 - \chi \left( \frac{d_F}{D} \right) \right]^2 + \\
 &\quad + \frac{u}{d_F} \left[ 1 - \chi \left( \frac{d_F}{D} \right) \right] F_\xi(\nabla d_F) \cdot \nabla u \quad (4.2)
 \end{aligned}$$

where last equality follows by  $F(\nabla d_F) = 1$ , the 1-homogeneity of  $F$  and property (2.4). Let us observe that, using the Divergence Theorem (in a general setting, contained for example in [A83]), we have

$$\begin{aligned}
 \int_{\Omega} \frac{u}{d_F} \left[ 1 - \chi \left( \frac{d_F}{D} \right) \right] F_\xi(\nabla d_F) \cdot \nabla u dx &= \\
 &= - \int_{\Omega} \frac{u^2}{2} \operatorname{div} \left( \left[ 1 - \chi \left( \frac{d_F}{D} \right) \right] \frac{F_\xi(\nabla d_F)}{d_F} \right) dx = \\
 &= \int_{\Omega} \frac{u^2}{2} \left\{ \left[ 1 - \chi \left( \frac{d_F}{D} \right) + \chi^2 \left( \frac{d_F}{D} \right) \right] \frac{F_\xi(\nabla d_F) \cdot \nabla d_F}{d_F^2} + \right. \\
 &\quad \left. - \left[ 1 - \chi \left( \frac{d_F}{D} \right) \right] \frac{\Delta_F d_F}{d_F} \right\} dx \geq \\
 &\geq \int_{\Omega} \frac{1}{2} \frac{u^2}{d_F^2} \left[ 1 - \chi \left( \frac{d_F}{D} \right) + \chi^2 \left( \frac{d_F}{D} \right) \right] dx, \quad (4.3)
 \end{aligned}$$

where last inequality follows using the condition  $-\Delta_F d_F \geq 0$ .

Integrating (4.2), and using (4.3) we easily get

$$\begin{aligned}
 \int_{\Omega} F(\nabla u)^2 dx &\geq \int_{\Omega} \frac{1}{4} \frac{u^2}{d_F^2} \times \\
 &\times \left\{ - \left[ 1 - \chi \left( \frac{d_F}{D} \right) \right]^2 + 2 - 2\chi \left( \frac{d_F}{D} \right) + 2\chi^2 \left( \frac{d_F}{D} \right) \right\} dx = \\
 &= \frac{1}{4} \int_{\Omega} \frac{u^2}{d_F^2} dx + \frac{1}{4} \int_{\Omega} \frac{u^2}{d_F^2} \chi^2 \left( \frac{d_F}{D} \right) dx,
 \end{aligned}$$

and the proof is completed.  $\square$

**Remark 4.1.** We observe that if  $\Omega$  is a convex domain in  $\mathbb{R}^n$ , arguing as in Remark 3.1, an inequality of the type (4.1) can be immediately obtained by using the following improved Hardy inequality involving  $d_\varepsilon$  contained in [BFT04]

$$\int_{\Omega} |\nabla u|^2 dx - \frac{1}{4} \int_{\Omega} \frac{|u|^2}{d_\varepsilon^2} dx \geq \frac{1}{4} \int_{\Omega} \frac{|u|^2}{d_\varepsilon^2} \left( \log \frac{D_0}{d_\varepsilon} \right)^{-2} dx, \quad (4.4)$$

where  $D_0 \geq e \cdot \sup d_\varepsilon(x, \partial\Omega)$  and  $u \in H_0^1(\Omega)$ . Obviously also in this case it is not possible to obtain the optimal constants.

**Corollary 4.1.** *Under the same assumptions of Theorem 4.1, the following anisotropic improved Hardy inequality holds*

$$\int_{\Omega} F(\nabla u)^2 dx - \frac{1}{4} \int_{\Omega} \frac{|u|^2}{d_F^2} dx \geq \frac{1}{4r_F^2} \int_{\Omega} |u|^2 dx, \quad (4.5)$$

where  $r_F$  is the anisotropic inradius defined in (2.12).

*Proof.* By Theorem 4.1, to prove (4.2) it is sufficient to show that

$$\int_{\Omega} \frac{|u|^2}{d_F^2} \left( \log \frac{D}{d_F} \right)^{-2} dx \geq \frac{1}{r_F^2} \int_{\Omega} |u|^2 dx.$$

This is a consequence of the monotonicity of the following function

$$f(t) = -t \log \left( \frac{t}{e \cdot r_F} \right), \quad 0 < t \leq r_F.$$

Indeed  $f$  is strictly increasing and its maximum is  $r_F$ . This concludes the proof.  $\square$

An immediate consequence of the previous result is contained in the following remark.

**Remark 4.2.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded convex domain. Then there exists a positive constant  $C(n) > 0$  such that for any  $u \in H_0^1(\Omega)$  we have

$$\int_{\Omega} F(\nabla u)^2 dx - \frac{1}{4} \int_{\Omega} \frac{|u|^2}{d_F^2} dx \geq C(n) |\Omega|^{-\frac{2}{n}} \int_{\Omega} |u|^2 dx.$$

## 5 OPTIMALITY OF THE CONSTANTS

Here we prove the optimality of the constants and of the exponent which appear in the Hardy inequality (4.1). More precisely, we prove the following result:

**Theorem 5.1.** *Let  $\Omega$  be a piecewise  $C^2$  domain of  $\mathbb{R}^n$ . Suppose that the following Hardy inequality holds:*

$$\int_{\Omega} F^2(\nabla u) \, dx - A \int_{\Omega} \frac{u^2}{d_F^2} \, dx \geq B \int_{\Omega} \frac{u^2}{d_F^2} \left( \log \frac{D}{d_F} \right)^{-\gamma} \, dx, \quad \forall u \in H_0^1(\Omega), \quad (5.1)$$

for some constants  $A > 0$ ,  $B \geq 0$ ,  $\gamma > 0$ , where  $D = e \cdot \sup\{d_F(x, \partial\Omega), x \in \Omega\}$ . Then:

- (T<sub>1</sub>)  $A \leq \frac{1}{4}$ ;
- (T<sub>2</sub>) If  $A = \frac{1}{4}$  and  $B > 0$ , then  $\gamma \geq 2$ ;
- (T<sub>3</sub>) If  $A = \frac{1}{4}$  and  $\gamma = 2$ , then  $B \leq \frac{1}{4}$ .

*Proof.* The proof is similar to the one obtained in the Euclidean case by [BFT04]. For the sake of completeness, we describe it in details. As before, let us denote by

$$X(t) = -\frac{1}{\log t}, \quad t \in ]0, 1[.$$

In order to prove the results we will provide a local analysis. Hence, we fix a Wulff shape  $\mathcal{W}_\delta$  of radius  $\delta$  centered at a point  $x_0 \in \partial\Omega$ . Being  $\partial\Omega$  piecewise smooth, we may suppose that for a sufficiently small  $\delta$ ,  $\partial\Omega \cap \mathcal{W}_\delta$  is  $C^2$ . Let now  $\varphi$  be a nonnegative cut-off function in  $C_0^\infty(\mathcal{W}_\delta \cap \Omega)$  such that  $\varphi(x) = 1$  for  $x \in \mathcal{W}_{\delta/2}$ . First of all, we prove some technical estimates which will be useful in the following. For  $\varepsilon > 0$  and  $\beta \in \mathbb{R}$  let us consider

$$J_\beta(\varepsilon) = \int_{\Omega} \varphi^2 d_F^{-1+2\varepsilon} X^{-\beta}(d_F/D) \, dx. \quad (5.2)$$

We split the proof in several claims.

**Claim 1.** The following estimates hold:

- (i)  $c_1 \varepsilon^{-1-\beta} \leq J_\beta(\varepsilon) \leq c_2 \varepsilon^{-1-\beta}$ , for  $\beta > -1$ , where  $c_1, c_2$  are positive constants independent of  $\varepsilon$ ;
- (ii)  $J_\beta(\varepsilon) = \frac{2\varepsilon}{\beta+1} J_{\beta+1}(\varepsilon) + O_\varepsilon(1)$ , for  $\beta > -1$ ;
- (iii)  $J_\beta(\varepsilon) = O_\varepsilon(1)$ , for  $\beta < -1$ .

*Proof of Claim 1.* By the coarea formula,

$$J_\beta(\varepsilon) = \int_0^\delta r^{-1+2\varepsilon} X^{-\beta}(r/D) \left( \int_{d_F=r} \frac{\varphi^2}{|\nabla d_F|} dH^{n-1} \right) dr$$

Being  $F(\nabla d_F) = 1$ , by (2.2),  $0 < \alpha_1 \leq |\nabla d_F| \leq \alpha_2$  and

$$0 < C_1 \leq \int_{d_F=r} \frac{\varphi^2}{|\nabla d_F|} dH^{n-1} \leq C_2.$$

Then if  $\beta < -1$ , (iii) easily follows. Moreover, if  $\beta > -1$ , performing the change of variables  $r = Ds^{1/\varepsilon}$ , (i) holds.

As regards (ii), let us observe that

$$\frac{d}{dr} X^\beta = \beta \frac{X^{\beta+1}}{r}.$$

Recalling that  $1 = F(\nabla d_F)F_\xi(\nabla d_F) \cdot \nabla d_F$ , then

$$\begin{aligned} (\beta + 1)J_\beta(\varepsilon) &= - \int_{\Omega} \varphi^2 d_F^{2\varepsilon} F(\nabla d_F) F_\xi(\nabla d_F) \cdot \nabla [X^{-\beta-1}(d_F/D)] dx = \\ &= \int_{\Omega} \operatorname{div} (\varphi^2 d_F^{2\varepsilon} F(\nabla d_F) F_\xi(\nabla d_F)) X^{-\beta-1}(d_F/D) dx = \\ &= 2 \int_{\Omega} \varphi d_F^{2\varepsilon} X^{-\beta-1}(d_F/D) F(\nabla d_F) F_\xi(\nabla d_F) \cdot \nabla \varphi dx + \\ &\quad + 2\varepsilon \int_{\Omega} \varphi d_F^{2\varepsilon-1} X^{-\beta-1}(d_F/D) dx + \\ &\quad + \int_{\Omega} \varphi^2 d_F^{2\varepsilon} X^{-\beta-1}(d_F/D) \Delta_F d_F dx = \\ &= O_\varepsilon(1) + 2\varepsilon J_{\beta+1}(\varepsilon). \end{aligned}$$

We explicitly observe that

$$\int_{\Omega} \varphi^2 d_F^{2\varepsilon} X^{-\beta-1}(d_F/D) \Delta_F d_F dx = O_\varepsilon(1)$$

being  $d_F$  a  $C^2$  function in a neighborhood of the boundary (see [CMo7]). Then (ii) holds.

In the next claim we estimate the left-hand side of (5.1) when  $u = U_\varepsilon$ , with

$$U_\varepsilon(x) = \varphi(x)w_\varepsilon(x), \quad w_\varepsilon(x) = d_F^{\frac{1}{2}+\varepsilon} X^{-\theta}(d_F(x)/D), \quad \frac{1}{2} < \theta < 1.$$

Let us define

$$\mathcal{Q}[U_\varepsilon] := \int_{\Omega} \left( F(\nabla U_\varepsilon)^2 - \frac{1}{4} \frac{U_\varepsilon^2}{d_F^2} \right) dx.$$

**Claim 2.** The following estimates hold:

$$\mathcal{Q}[U_\varepsilon] \leq \frac{\theta}{2} J_{2\theta-2}(\varepsilon) + O_\varepsilon(1), \quad \text{as } \varepsilon \rightarrow 0; \quad (5.3)$$

$$\int_{W_\delta \cap \Omega} F(\nabla U_\varepsilon)^2 dx \leq \frac{1}{4} J_{2\theta}(\varepsilon) + O_\varepsilon(\varepsilon^{1-2\theta}), \quad \text{as } \varepsilon \rightarrow 0. \quad (5.4)$$

*Proof of Claim 2.* The convexity of  $F$  implies that

$$F(\xi + \eta)^2 \leq F(\xi)^2 + 2F(\xi)F(\eta) + F(\eta)^2, \quad \forall \xi, \eta \in \mathbb{R}^n.$$

Hence by the homogeneity of  $F$ ,

$$\begin{aligned} \int_{\Omega} F(\nabla U_{\varepsilon})^2 dx &\leq \\ &\leq \int_{W_{\delta} \cap \Omega} \varphi^2 F^2(\nabla w_{\varepsilon}) dx + \int_{W_{\delta} \cap \Omega} w_{\varepsilon}^2 F^2(\nabla \varphi) dx + \\ &\quad + \int_{W_{\delta} \cap \Omega} 2\varphi w_{\varepsilon} F(\nabla \varphi) F(\nabla w_{\varepsilon}) dx = \\ &= \int_{W_{\delta} \cap \Omega} \varphi^2 d_F^{2\varepsilon-1} X^{-2\theta}(d_F/D) \left( \varepsilon + \frac{1}{2} - \theta X(d_F/D) \right)^2 dx + I_1 + I_2. \end{aligned}$$

As matter of fact,

$$I_2 \leq C \int_{W_{\delta} \cap \Omega} d_F^{2\varepsilon} X^{-2\theta}(d_F/D) dx = O_{\varepsilon}(1);$$

similarly, also  $I_1 = O_{\varepsilon}(1)$ . Then

$$\begin{aligned} Q[U_{\varepsilon}] &\leq \\ &\leq \int_{W_{\delta} \cap \Omega} \varphi^2 d_F^{2\varepsilon-1} X^{-2\theta}(d_F/D) \left[ \left( \varepsilon + \frac{1}{2} - \theta X(d_F/D) \right)^2 - \frac{1}{4} \right] dx \\ &\quad + O_{\varepsilon}(1) = \\ &\leq \int_{W_{\delta} \cap \Omega} \varphi^2 d_F^{2\varepsilon-1} X^{-2\theta}(d_F/D) (\varepsilon - \theta X(d_F/D))^2 dx + \\ &\quad + \int_{W_{\delta} \cap \Omega} \varphi^2 d_F^{2\varepsilon-1} X^{-2\theta}(d_F/D) (\varepsilon - \theta X(d_F/D)) + O_{\varepsilon}(1) = \\ &= \alpha_1 + \alpha_2 + O_{\varepsilon}(1). \quad (5.5) \end{aligned}$$

Using (ii) of Claim 1 with  $\beta = -1 + 2\theta$  we get

$$\alpha_2 = O_{\varepsilon}(1) \quad \varepsilon \rightarrow 0. \quad (5.6)$$

As regards  $\alpha_1$ , similarly, applying (ii) of Claim 1 with  $\beta = 2\theta - 1$  in the first time and  $\beta = 2\theta - 2$  in the second time we obtain

$$\alpha_1 = \frac{\theta}{2} \int_{W_{\delta}} \varphi^2 d_F^{2\varepsilon-1} X^{2-2\theta}(d_F/D) dx + O_{\varepsilon}(1). \quad (5.7)$$

Then (5.3) follows by (5.5), (5.6), (5.7) and (5.2). Finally observing that

$$\int_{W_{\delta} \cap \Omega} F(\nabla U_{\varepsilon})^2 dx = Q[U_{\varepsilon}] + \frac{1}{4} J_{2\theta}(\varepsilon), \quad (5.8)$$

then the inequality (5.4) follows from (5.3) and (ii) of Claim 1.

Now we are in position to conclude the proof of the Theorem.

Since inequality (5.1) holds for any  $u \in H_0^1(\Omega)$  we take as test function  $U_\varepsilon$ . Then by (5.4) and (i) of Claim 1 we have

$$A \leq \frac{1}{J_{2\theta}(\varepsilon)} \int_{W_\delta \cap \Omega} F(\nabla U_\varepsilon)^2 dx \leq \frac{1}{4} + O_\varepsilon(\varepsilon).$$

Letting  $\varepsilon \rightarrow 0$  we obtain (T<sub>1</sub>).

In order to prove (T<sub>2</sub>) we put  $A = \frac{1}{4}$  and reasoning by contradiction we assume that  $\gamma < 2$ . As before by (5.3) and (i) of Claim 1, we have

$$0 < B \leq \frac{Q[U_\varepsilon]}{J_{2\theta-\gamma}(\varepsilon)} \leq C \frac{\varepsilon^{1-2\theta}}{\varepsilon^{\gamma-1-2\theta}} = C\varepsilon^{2-\gamma} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

which is a contradiction and then  $\gamma \geq 2$ .

To conclude the proof of the Theorem we just have to prove (T<sub>3</sub>).

If  $A = \frac{1}{4}$  and  $\gamma = 2$  then by (5.3) we have

$$B \leq \frac{Q[U_\varepsilon]}{J_{2\theta-2}(\varepsilon)} \leq \frac{\frac{\theta}{2} J_{2\theta-2}(\varepsilon) + O_\varepsilon(1)}{J_{2\theta-2}(\varepsilon)}.$$

Then by assumption on  $\theta$  and (i) of Claim 1, letting  $\varepsilon \rightarrow 0$  we get

$$B \leq \frac{\theta}{2}.$$

Hence (T<sub>3</sub>) follows by letting  $\theta \rightarrow \frac{1}{2}$ . □

**Remark 5.1.** We stress that Theorem 5.1 assures that the involved constants in (4.1) are optimal, and also in the anisotropic Hardy inequality (3.1). Actually, the presence of the remainder term in (4.1) guarantees that the constant  $\frac{1}{4}$  in (3.1) is not achieved.

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